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By Philip R. Nachtsheim

SUMMARY

The stability of free-convection boundary-layer flows is investigated by numerical integration of the disturbance differential equations. Stability calculations are carried out for Prandtl numbers of 0.733 (air) and 6.7 (water) with and without temperature fluctuations. Results presented for these four cases consist of eigenvalues (phase velocity, wave number, and Reynolds number), eigenfunctions, and energy distribution curves for neutrally stable disturbances. Tabulations of the basic velocity and temperature profiles for a Prandtl number of 6.7 are also included.

When temperature fluctuations are included, a mode of instability is found in which the primary source of energy for the disturbance motion arises from the interaction of buoyancy forces with velocity fluctuations. The present stability results are compared with available theoretical and experimental results.

INTRODUCTION

It is well known that boundary-layer flows are unstable under certain conditions in the sense that a small disturbance imposed on the basic flow can grow indefinitely in time. This instability is related to the shearing motion of the basic flow and existence of a mechanism for transferring energy from the basic flow to the disturbance motion via the Reynolds stress. It is also well known that a static fluid in a gravitational field in which a constant temperature gradient is maintained is unstable under certain conditions. This thermal instability arises when the gravity vector has a component parallel to the temperature gradient. In this case a supply of energy for the disturbance flow comes from the potential energy of the fluid particles in the static configuration. Both of these stability problems have received extensive, but separate, treatments in the literature.

The problem of the stability of free-convection boundary-layer flows is an interesting combination of the problems of boundary-layer stability and thermal stability. Since the free-convection flow is a shearing motion, the problem of its stability contains all the features of the boundary-layer stability problem. However, when a temperature disturbance is imposed on the free-convection boundary-layer flow, the essential feature of the thermal stability problem is introduced, and there is a possibility of transferring energy to the disturbance flow if there is a correlation between the disturbance velocity component along

the plate and the disturbance temperature gradient along the plate.

The determination of the stability characteristics of a given free-convection boundary-layer flow requires the determination of the eigenvalues occurring in a system of complex differential equations of sixth order. This system of equations differs from the well-known Orr-Sommerfeld fourth-order equation, which applies to ordinary boundary-layer flows in that the sixth-order system takes account of the interaction of the gravity force with density fluctuations.

The effect of temperature fluctuations has been neglected in all previous investigations of the stability of free-convection boundary-layer flows. Reference 1 showed that the neglect of temperature fluctuations is justified if the Reynolds number is sufficiently large; however, the Reynolds numbers at which finite disturbances have been observed experimentally in free-convection flows are not extremely large numbers.

With the neglect of temperature fluctuations the disturbance differential equation for free-convection boundary-layer flows reduces to the Orr-Sommerfeld equation. The asymptotic techniques that were developed to solve the Orr-Sommerfeld equation were used in references 1 and 2 to solve the free-convection boundary-layer stability problem.

In asymptotic methods, the critical layer plays a central role in the analysis. The critical layer is located at that point in the boundary layer where the phase velocity of an assumed disturbance wave matches the velocity of the basic flow. For conventional velocity profiles, linearly independent solutions of the Orr-Sommerfeld equation are often obtained in the form of expansions about the critical layer. Since the velocity in the free-convection flow is zero at the plate and at large distances from the plate, each value of velocity is attained twice, once when the velocity is increasing and once when the velocity is decreasing; hence, for these profiles there will be two critical layers or none. In references 1 and 2, account is taken of the presence of two critical layers by obtaining solutions to the differential equation by means of expansions about each critical layer. However, minimum critical Reynolds numbers, that is, the Reynolds numbers below which all disturbances should be damped, were obtained that were greater than the Reynolds numbers at which finite disturbances were observed experimentally (refs. 1 and 2).

In order to avoid the problem of expanding solutions about two critical layers, other investigators have used direct numerical methods to solve the Orr-Sommerfeld equation. A numerical method to determine the stability characteristics of a transverse velocity profile near the stagnation point of a sweptback wing was developed in references 3 and 4. This method consists essentially of using step-by-step integration to find two linearly independent solutions of the differential equation that satisfy the boundary conditions at the wall. A suitable linear combination of these solutions is then sought by a trial-and-error method in order to match the known free-stream solution at the edge of the boundary layer. In reference 5, the method developed in reference 6 was applied to determine the stability characteristics of a free-convection velocity profile without temperature fluctuations. A minimum critical Reynolds number below the

Reynolds number at which finite disturbances were observed experimentally was obtained in reference 5. However, none of these methods made use of any of the analytic properties of the Orr-Sommerfeld differential equation in order to simplify the numerical calculations.

In the present numerical method, use is made of the property of the disturbance differential equations that solutions of the differential equations depend analytically on the parameters appearing in the equations. This method of solution, unlike the asymptotic methods, does not rely on the existence of a critical layer.

The basic idea of the present method is to integrate the differential equations by a step-by-step method for guessed values of the eigenvalues and starting values as if they were nonlinear equations. The boundary-value problem is treated as an initial value problem for which not all the proper starting values are known so as to satisfy conditions at the other boundary. Equations involving the dependent variables are formulated and evaluated at the edge of boundary layer so that the zeros of these equations indicate that the numerical solution matches the known free-stream solution. The Newton-Raphson second-order process is used to obtain corrections to the eigenvalues and the starting values in order to satisfy boundary conditions. The iteration process is continued until the boundary conditions are satisfied.

This general method outlined is described in reference 7. The purpose of this report is to apply the method to the solution of the free-convection boundary-layer stability differential equations and to show how the analytic properties of the differential equations can be used to simplify the application of the general method.

DISTURBANCE EQUATIONS AND THEIR SOLUTIONS

Disturbance Equations

The disturbance equations for a parallel flow as given in reference 1 are as follows:

$$\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi + s' = i\alpha\text{Re}\left[(F' - c)(\phi'' - \alpha^2\phi) - F'''\phi\right] \quad (1)$$

$$s'' - \alpha^2s = i\alpha\text{RePr}\left[(F' - c)s - \phi H'\right] \quad (2)$$

(Symbols are defined in appendix A.)

Equations similar to these were also derived in reference 2 by means of a different nondimensionalizing procedure. A brief derivation of equations (1) and (2) is given in appendix B. The effect of the temperature fluctuations appears in the function s , which couples equations (1) and (2). If s' is set equal to zero, equation (1) reduces to the well-known Orr-Sommerfeld equation. The boundary conditions for equations (1) and (2), in the case of an isothermal plate, are

$$\eta = 0 \quad \varphi = \varphi' = s = 0 \quad (3)$$

$$\eta \rightarrow \infty \quad \varphi, \varphi', s \rightarrow 0 \quad (4)$$

The boundary condition that the temperature disturbance amplitude s vanishes at the surface is a reasonable one for metallic plates immersed in a fluid such as air or water, where the plate materials are highly conductive compared to the fluid.

Replacement of Boundary Conditions at Infinity

The boundary conditions at infinity can be replaced by different conditions, which are to be satisfied at a finite distance from the plate. The new conditions are to be applied at the edge of the boundary layer at $\eta = \eta_e$, where the values of F' , F''' , and H' are nearly at their asymptotic values. For free-convection profiles F' , F''' , and H' approach zero at the edge of the boundary layer; therefore, for $\eta \geq \eta_e$ the coefficients in equations (1) and (2) can be treated as constants, and equations (1) and (2) reduce to

$$\varphi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi + s' = -i\alpha\text{Rec}(\varphi'' - \alpha^2\varphi) \quad (5)$$

$$s'' - \alpha^2s = -i\alpha\text{RecPr}s \quad (6)$$

The general solution of the system of equations (5) and (6) is

$$\begin{aligned} \varphi = & c_1 \exp(\alpha\eta) + c_2 \exp(-\alpha\eta) + c_3 \exp(\beta\eta) \\ & + c_4 \exp(-\beta\eta) + c_5 \exp(\gamma\eta) + c_6 \exp(-\gamma\eta) \end{aligned} \quad (7)$$

$$s = \frac{-c_5(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)}{\gamma} \exp(\gamma\eta) + c_6 \frac{(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)}{\gamma} \exp(-\gamma\eta) \quad (8)$$

where c_1, c_2, \dots, c_6 are arbitrary constants and where β and γ , respectively, are the roots with positive real parts of

$$\beta^2 = \alpha^2 - i\alpha\text{Rec} \quad (9)$$

$$\gamma^2 = \alpha^2 - i\alpha\text{RecPr} \quad (10)$$

The task remains of eliminating the arbitrary constants in equations (7) and (8) in order to obtain a set of linear and homogeneous relations in the dependent variables and their derivatives, which are to be satisfied at the edge of the boundary layer. The elimination of the arbitrary constants is carried out in appendix C, where it is shown that solutions of equations (1) and (2), which decay exponentially as the distance from the plate increases indefinitely, are characterized by three conditions that must be satisfied at the edge of the

boundary layer, $\eta = \eta_e$. These conditions are

$$s' + \gamma s = 0 \quad (11)$$

$$\varphi''' - \alpha^2 \varphi' + \beta(\varphi'' - \alpha^2 \varphi) + \frac{\gamma}{\gamma + \beta} s = 0 \quad (12)$$

$$\varphi''' + \alpha \varphi'' - \beta^2(\varphi' + \alpha \varphi) + \frac{\gamma}{\gamma + \alpha} s = 0 \quad (13)$$

Solution of Eigenvalue Problem

The approach to the eigenvalue problem for fixed Pr used herein is to fix α and Re and then to find values of $c = c_r + ic_i$ (eigenvalues) for which equations (1) and (2) have solutions (eigenfunctions) that satisfy the boundary conditions (eqs. (3) and (11) to (13)).

The eigenvalue c and the corresponding eigenfunctions are obtained by treating equations (1) and (2) as if they were nonlinear equations. A trial solution is obtained by step-by-step numerical integration of the differential equations starting at $\eta = 0$ for assumed starting values and an assumed value of c . Equations (11) to (13) are evaluated at the edge of the boundary layer. If the boundary conditions are not satisfied, the starting values and c are adjusted, and another trial is made. For the assumed values of the parameters, denote the values taken on by the expressions on the left side of equations (11) to (13) by

$$A_1 = s' + \gamma s \quad (14)$$

$$A_2 = \varphi''' - \alpha^2 \varphi' + \beta(\varphi'' - \alpha^2 \varphi) + \frac{\gamma}{\gamma + \beta} s \quad (15)$$

$$A_3 = \varphi''' + \alpha \varphi'' - \beta^2(\varphi' + \alpha \varphi) + \frac{\gamma}{\gamma + \alpha} s \quad (16)$$

It is desired to find the eigenfunctions and the values of the parameters that will simultaneously cause A_1 , A_2 , and A_3 to take on the value zero. In order to calculate A_1 , A_2 , and A_3 , starting values $\varphi(0)$, $\varphi'(0)$, $\varphi''(0)$, $\varphi'''(0)$, $s(0)$, and $s'(0)$ have to be assigned as well as a value for c . The boundary conditions at the plate are satisfied by taking $\varphi(0) = \varphi'(0) = s(0) = 0$. Since the differential equations (1) and (2) are linear and homogeneous and the boundary conditions are homogeneous, it is permissible to take $\varphi''(0)$ equal to some convenient fixed value. This value sets the magnitude of the linear oscillation. The remaining unknown starting values, $\varphi'''(0)$ and $s'(0)$, are denoted by a and b , respectively.

Newton-Raphson method is used to obtain the zeros of equations (14) to (16). If the chosen values a , b , and c produce a solution $\varphi(\eta; a, b, c)$ and $s(\eta; a, b, c)$ that approximately satisfies equations (14) to (16), that is, actually

leads to values of A_1 , A_2 , and A_3 close to zero, then a better approximation is obtained by starting with $a + \Delta a$, $b + \Delta b$, and $c + \Delta c$ instead of a , b , and c , where Δa , Δb , and Δc are obtained as solutions of

$$0 = A_1(a, b, c) + \frac{\partial A_1}{\partial a} \Delta a + \frac{\partial A_1}{\partial b} \Delta b + \frac{\partial A_1}{\partial c} \Delta c \quad (17)$$

$$0 = A_2(a, b, c) + \frac{\partial A_2}{\partial a} \Delta a + \frac{\partial A_2}{\partial b} \Delta b + \frac{\partial A_2}{\partial c} \Delta c \quad (18)$$

$$0 = A_3(a, b, c) + \frac{\partial A_3}{\partial a} \Delta a + \frac{\partial A_3}{\partial b} \Delta b + \frac{\partial A_3}{\partial c} \Delta c \quad (19)$$

The quantities A_1 , A_2 , and A_3 and the partial derivatives evaluated at $\eta = \eta_e$ can be determined by integrating the differential equations (1) and (2) by a step-by-step method. The partial derivatives in equations (17) to (19) may be given either by a finite difference approximation or from equations obtained by partial differentiation of equations (1) and (2) with respect to a , b , and c . (The coefficients of eqs. (1) and (2) are analytic functions of η and depend analytically on the parameters α , Re , and c . Therefore, the solutions of eqs. (1) and (2) have the same analytical properties and have the required partial derivatives.) In the former method four integrations of equations (1) and (2) are carried out, one with a basic set a , b , and c , and three more, each with appropriate small increments on a , b , and c . Partial differentiation gives the same information with one integration. However, in this second method, three extra sets of differential equations have to be integrated.

The equations for the derivatives with respect to a are as follows:

$$\varphi_a'''' - 2\alpha^2 \varphi_a'' + \alpha^4 \varphi_a + s' = i\alpha \text{Re} [(F' - c)(\varphi_a'' - \alpha^2 \varphi_a) - F''' \varphi_a] \quad (20)$$

$$s_a'' - \alpha^2 s_a = i\alpha \text{RePr} [(F' - c)s_a - \varphi_a H'] \quad (21)$$

with the initial conditions

$$\eta = 0 \quad \varphi_a = \varphi_a' = \varphi_a'' = s_a = s_a' = 0 \quad \varphi_a''' = 1 \quad (22)$$

The equations for the derivatives with respect to b are as follows:

$$\varphi_b'''' - 2\alpha^2 \varphi_b'' + \alpha^4 \varphi_b + s_b' = i\alpha \text{Re} [(F' - c)(\varphi_b' - \alpha^2 \varphi_b) - F''' \varphi] \quad (23)$$

$$s_b'' - \alpha^2 s_b = i\alpha \text{RePr} [(F' - c)s_b - \varphi_b H'] \quad (24)$$

with the initial conditions

$$\eta = 0 \quad \varphi_b = \varphi'_b = \varphi''_b = \varphi'''_b = s_b = 0 \quad s'_b = 1 \quad (25)$$

and the equations for the derivatives with respect to c are as follows:

$$\varphi_c''' - 2\alpha^2\varphi_c'' + \alpha^4\varphi_c + s'_c = i\alpha\text{Re}[(F' - c)(\varphi_c'' - \alpha^2\varphi_c) - F''' \varphi_c - (\varphi'' - \alpha^2\varphi)] \quad (26)$$

$$s_c'' - \alpha^2 s_c = i\alpha\text{RePr}[(F' - c)s_c - \varphi_c H' - s] \quad (27)$$

with the initial conditions

$$\eta = 0 \quad \varphi_c = \varphi'_c = \varphi''_c = \varphi'''_c = s_c = s'_c = 0 \quad (28)$$

In these equations the subscripts a , b , and c denote partial differentiation with respect to $a \equiv \varphi'''(0)$, $b \equiv s'(0)$, and c , respectively. For example, $\varphi_a = (\partial/\partial a)\varphi(\eta; a, b, c)$ gives the rate of change of φ with respect to a with b , c , and η held constant as a function of η . The partial derivatives of the A 's appearing in equations (17) to (19) can now be evaluated. Note that the boundary conditions (eqs. (14) to (16)) involve c , the eigenvalue, through the intermediate variables β and γ defined by equations (9) and (10). Utilizing equations (14) to (16) results in

$$\frac{\partial A_1}{\partial a} = s'_a + \gamma s_a \quad (29)$$

$$\frac{\partial A_1}{\partial b} = s'_b + \gamma s_b \quad (30)$$

$$\frac{\partial A_1}{\partial c} = s'_c + \gamma s_c - \frac{i\alpha\text{RePr}}{2} \frac{1}{\gamma} s \quad (31)$$

$$\frac{\partial A_2}{\partial a} = \varphi_a''' - \alpha^2\varphi_a'' + \beta(\varphi_a'' - \alpha^2\varphi_a) + \frac{\gamma}{\gamma + \beta} s_a \quad (32)$$

$$\frac{\partial A_2}{\partial b} = \varphi_b''' - \alpha^2\varphi_b'' + \beta(\varphi_b'' - \alpha^2\varphi_b) + \frac{\gamma}{\gamma + \beta} s_b \quad (33)$$

$$\begin{aligned} \frac{\partial A_2}{\partial c} = & \varphi_c''' - \alpha^2\varphi_c'' + \beta(\varphi_c'' - \alpha^2\varphi_c) + \frac{\gamma}{\gamma + \beta} s_c \\ & - \frac{i\alpha\text{Re}}{2\beta} \left[(\varphi'' - \alpha^2\varphi) + \frac{\alpha^2(\text{Pr} - 1)}{\gamma} \frac{s}{(\gamma + \beta)^2} \right] \end{aligned} \quad (34)$$

$$\frac{\partial A_3}{\partial a} = \varphi_a''' + \alpha \varphi_a'' - \beta^2(\varphi_a' + \alpha \varphi_a) + \frac{\gamma}{\gamma + \alpha} s_a \quad (35)$$

$$\frac{\partial A_3}{\partial b} = \varphi_b''' + \alpha \varphi_b'' - \beta^2(\varphi_b' + \alpha \varphi_b) + \frac{\gamma}{\gamma + \alpha} s_b \quad (36)$$

$$\frac{\partial A_3}{\partial c} = \varphi_c''' + \alpha \varphi_c'' - \beta^2(\varphi_c' + \alpha \varphi_c) + \frac{\gamma}{\gamma + \alpha} s_c + iaRe \left[(\varphi' + 2\varphi - \frac{\alpha Pr}{2\gamma} \frac{s}{(\gamma + \alpha)^2}) \right] \quad (37)$$

For a given point (α, Re) in the α, Re -plane, the procedure outlined should converge to an eigenvalue c .

Criterion for Convergence to an Eigenvalue

The procedure outlined previously for finding the eigenvalue for a given point in the α, Re -plane was programed for solution by using complex arithmetic on the IBM 7090 located at the Lewis Research Center. The numerical integration was done with the Runge-Kutta method, which gives fourth-order accuracy. Convergence to an eigenvalue was established by requiring that

$|A_1|^2 + |A_2|^2 + |A_3|^2 < 5 \times 10^{-6}$ or requiring that $|\Delta a|$, $|\Delta b|$, and $|\Delta c|$ all be less than or equal to 5×10^{-6} .

The step size was externally controlled and set such that there was no significant change in the eigenvalue when the example was rerun with a step size equal to one-half of the original value.

In order to fix the size of the solution, the eigenfunctions were normalized after each integration so that φ was set equal to unity at the edge of the boundary layer.

The effect on the results of the choice of the value of η for the end of the range of integration was examined. For air η_e was taken to be 6, and for water η_e was taken to be 5. Variations of η_e from these values for both cases produced little or no changes in the values of the eigenvalues. As a matter of fact, η_e was taken to be 3 for exploratory runs that resulted in good approximations to the eigenvalues.

Determination of Neutral Curve

The results of the stability analysis are usually displayed in an α, Re -diagram on which the curve $c_1 = 0$ is drawn. On this curve Re can be considered a function of α . The minimum value of Re for points on this curve is called the minimum critical Reynolds number. The disturbances corresponding to points in the α, Re -plane with Reynolds numbers below the critical value will be damped.

The procedure used herein to find the critical Reynolds number consisted in holding Re constant and plotting the imaginary part c_i of the established eigenvalues against α . The values of α for which $c_i = 0$ could be read from such a graph establishing points on the neutral curve. For those values of Re below the critical Reynolds number, the plot of c_i against α would not give any $c_i = 0$. However, after the critical Reynolds number had been bracketed, α was held constant, and successive linear interpolations led to the critical Reynolds number. Other points on the neutral curve were obtained by plotting c_i against either α or Re and holding the other constant, depending on which choice proved to be more convenient. Although this procedure is straightforward and easy to apply, it is an indirect way to establish points on the neutral curve. This disadvantage can be attributed to the choice of the complex number c as an eigenvalue. The advantage of using c as an eigenvalue will become apparent after examining the differential equations to be integrated.

Fixing α and Re and determining the relation between the four parameters α , Re , and (c_r, c_i) by satisfying the boundary conditions (eqs. (11) to (13)) correspond to the choice of (c_r, c_i) as the unknown eigenvalue pair and lead to a system of 24 first-order complex differential equations to be integrated. Since points on the neutral curve ($c_i = 0$) are of primary interest, another choice is to set $c_i = 0$ and then attempt to find the proper relation among the parameters α , Re , and c_r by solving equations (17) to (19) with $c_i = 0$ and $\Delta c_i = 0$. Setting $c_i = 0$ leads to the solution of six real equations in five real unknowns. In general, this set of equations will not be consistent unless the parameters α , Re , and c_r are chosen properly. However, the proper value of these parameters is the information being sought. Since the value of c_i has been specified, it is no longer permissible to fix both α and Re , and the variation of one of these parameters has to be allowed for in order to provide a consistent set of six real equations in six real unknowns. However, the introduction of this additional unknown requires the integration of an additional set of six first-order complex differential equations.

Hence, the choice of (c_r, c_i) as the unknown eigenvalue pair among the parameters α , Re , and (c_r, c_i) leads to the fewest number of differential equations to be integrated. This certainly justifies the choice of c as an eigenvalue for the case when the differential equations are solved using complex arithmetic. If the system of differential equations is written in real form, the number of equations to be integrated adds up to 84. However, 72 of these equations involved partial derivatives with respect to the starting values and the eigenvalue c . The number of equations to be integrated can be reduced if use is made of the property that the solutions of equations (1) and (2) are analytic functions of η and depend analytically on the parameters α , Re , and c . This means that the information provided by the integration of the 72 real equations given by equations (20) to (28) can be obtained by the integration of only 36 real equations and the use of the Cauchy-Riemann relations, thus giving a total of 48 real equations to be integrated. For example, the rate of change of ϕ with respect to c_i is given in terms of derivatives with respect to c_r by relations of the form: $\partial\phi_i/\partial c_i = \partial\phi_r/\partial c_r$ and $\partial\phi_r/\partial c_i = -\partial\phi_i/\partial c_r$. Insisting that $\Delta c_i = 0$ would amount to rejecting the information furnished by the Cauchy-Riemann relations.

Another justification for the choice of (c_r, c_i) as the unknown eigenvalue pair, thereby allowing c_i to take on nonzero values, is that this choice may lead to a solution of the eigenvalue problem, whereas setting $c_i = 0$ at the outset may eliminate any solutions. For example, the eigenvalue problem in the case of plane Couette flow has no solution for $c_i = 0$.

ENERGY BALANCE OF DISTURBANCE MOTION

After the eigenvalue problem has been solved and the eigenfunctions have been determined, the energy balance of the disturbance motion can be computed.

The time rate of increase of the disturbance kinetic energy per unit of volume of a fluid particle that moves with the basic flow is

$$\frac{D}{D\tau} \left[\frac{\rho}{2} (\tilde{u}^2 + \tilde{v}^2) \right] = \left(\frac{\partial}{\partial \tau} + U \frac{\partial}{\partial x} \right) \left[\frac{\rho}{2} (\tilde{u}^2 + \tilde{v}^2) \right] \quad (38)$$

An equation governing the time rate of increase of the kinetic energy can be obtained by the same technique as used in reference 8 for ordinary boundary layers, that is, by multiplying equation (B6) by \tilde{u} and equation (B7) by \tilde{v} , adding these equations, and using equation (B5) to simplify. This procedure yields

$$\begin{aligned} \frac{D}{D\tau} \left[\frac{\rho}{2} (\tilde{u}^2 + \tilde{v}^2) \right] = & -\rho \tilde{u} \tilde{v} \frac{dU}{dy} - \left(\frac{\partial \tilde{p} \tilde{u}}{\partial x} + \frac{\partial \tilde{p} \tilde{v}}{\partial y} \right) + \rho g \beta \tilde{u} \tilde{t} \\ & - \rho \nu \left(\frac{\partial \tilde{v}}{\partial x} - \frac{\partial \tilde{u}}{\partial y} \right)^2 + \rho \nu \left[\frac{\partial}{\partial x} (\tilde{v} \tilde{\xi}) - \frac{\partial}{\partial y} (\tilde{u} \tilde{\xi}) \right] \end{aligned} \quad (39)$$

where $\tilde{\xi} = (\partial \tilde{v} / \partial x) - (\partial \tilde{u} / \partial y)$.

Of course, all disturbance quantities are taken to be real in the energy balance equation (39). The terms in this equation are to be integrated with respect to y from $y = 0$ to $y = \infty$ and with respect to x over a wavelength $\bar{\lambda}$ of the disturbance in order to obtain the growth of kinetic energy per unit of time depth and area. After integration the second and last terms on the right side of equation (39) vanish since \tilde{u} and \tilde{v} vanish for $y = 0$ and $y = \infty$ and have the period $\bar{\lambda}$ with respect to x . The equation governing the time rate of increase of kinetic energy of the disturbance motion is

$$\frac{D}{D\tau} \int_{x=0}^{\bar{\lambda}} \int_{y=0}^{\infty} dx dy \left[\frac{\rho}{2} (\tilde{u}^2 + \tilde{v}^2) \right] = \rho \int_{x=0}^{\bar{\lambda}} \int_{y=0}^{\infty} dx dy \left[-\tilde{u} \tilde{v} \frac{dU}{dy} + g \beta \tilde{u} \tilde{t} - \nu \left(\frac{\partial \tilde{v}}{\partial x} - \frac{\partial \tilde{u}}{\partial y} \right)^2 \right] \quad (40)$$

The first term in the integrand of the right side of equation (40) gives the rate at which the basic flow, in virtue of its shearing motion, is working against the Reynolds stresses arising from the disturbance, the last term represents the dissipation of the disturbance motion, and the second term represents the work done by the buoyancy force.

The reference quantities introduced in appendix B are used to transform equation (40) to the following dimensionless form:

$$\frac{D}{D\tau} KE = \int_{\xi=0}^{\lambda} \int_{\eta=0}^{\infty} d\xi d\eta \left[- \frac{\tilde{u}}{U^*} \frac{\tilde{v}}{U^*} F'' + \frac{1}{Re} \frac{\tilde{t}}{\Delta T} \frac{\tilde{u}}{U^*} - \frac{1}{Re} \left(\frac{\partial}{\partial \xi} \frac{\tilde{v}}{U^*} - \frac{\partial}{\partial \eta} \frac{\tilde{u}}{U^*} \right)^2 \right] \quad (41)$$

where

$$KE = \int_{\xi=0}^{\lambda} \int_{\eta=0}^{\infty} d\xi d\eta \left(\frac{1}{2} \rho \frac{\tilde{u}^2 + \tilde{v}^2}{U^{*2}} \right) \quad (42)$$

The disturbance quantities in the integrand on the right side of equation (41) are given by the following relations:

$$\frac{\tilde{u}}{U^*} = \text{Re} \left\{ \varphi'(\eta) \exp [i\alpha(\xi - c\tau)] \right\} \quad (43)$$

$$\frac{\tilde{v}}{U^*} = \text{Re} \left\{ -i\alpha\varphi(\eta) \exp [i\alpha(\xi - c\tau)] \right\} \quad (44)$$

$$\frac{\tilde{t}}{\Delta T} = \text{Re} \left\{ s(\eta) \exp [i\alpha(\xi - c\tau)] \right\} \quad (45)$$

where Re denotes the real part of the complex quantity.

The integrals appearing in equation (41) are evaluated through substitution of equations (43) to (45) followed by integration with respect to ξ . It is found that, for neutral disturbances,

$$\begin{aligned} \frac{1}{\pi} \frac{D}{D\tau} KE = & - \int_0^{\infty} d\eta (\varphi_r' \varphi_i - \varphi_i' \varphi_r) F'' + \frac{1}{\alpha Re} \int_0^{\infty} d\eta (s_r \varphi_r' + s_i \varphi_i') \\ & - \frac{1}{\alpha Re} \int_0^{\infty} d\eta \left[(\alpha^2 \varphi_r - \varphi_r'')^2 + (\alpha^2 \varphi_i - \varphi_i'')^2 \right] \end{aligned} \quad (46)$$

The following designations are used to identify the integrands in equation (46):

$$e_{Re} = -(\varphi_r' \varphi_i - \varphi_i' \varphi_r) F'' \quad (47)$$

$$e_B = \frac{1}{\alpha Re} (s_r \varphi_r' + s_i \varphi_i') \quad (48)$$

$$e_D = -\frac{1}{\alpha Re} \left[(\alpha^2 \varphi_r - \varphi_r'')^2 + (\alpha^2 \varphi_i - \varphi_i'')^2 \right] \quad (49)$$

For neutral disturbances the energy balance gives

$$\int_0^\infty d\eta e_{Re} + \int_0^\infty d\eta e_B + \int_0^\infty d\eta e_D = 0 \quad (50)$$

The satisfaction of equation (50) provides a check on the solution of the eigenvalue problem.

BASIC VELOCITY AND TEMPERATURE PROFILES FOR WATER

The $Pr = 0.733$ velocity and temperature profiles are tabulated in reference 9. However, for the case $Pr = 6.7$ results were not available. In order to obtain the $Pr = 6.7$ velocity and temperature profiles, the free-convection boundary value problem was solved by treating it as an initial value problem as described in the INTRODUCTION. However, in this case there is no eigenvalue in the equations, rather the proper starting values have to be determined in order to satisfy conditions at infinity. The boundary value problem as given in reference 9 consists of solving

$$F''' + 3FF'' - 2(F')^2 + H = 0 \quad (51)$$

$$H'' + 3PrFH' = 0 \quad (52)$$

subject to the boundary conditions

$$F(0) = F'(0) = 0 \quad H(0) = 1 \quad (53)$$

$$F'(\infty) = H(\infty) = 0 \quad (54)$$

The tabulation of the $Pr = 6.7$ velocity profile is given in table I and is shown in figure 1 along with the $Pr = 0.733$ profile.

With the velocity and temperature profiles for both air and water available, the stability of these two free-convection boundary-layer flows can be investigated.

RESULTS AND DISCUSSION

Stability calculations were carried out for the free convection of air

($Pr = 0.733$) and water ($Pr = 6.7$) with and without temperature fluctuations.

Stability Results for Air Without Temperature Fluctuations

Neglecting temperature fluctuations amounts to solving the ordinary boundary-layer stability problem with the prescribed free-convection velocity profile inserted in the Orr-Sommerfeld equation, which is equation (1) with s' set equal to zero.

The results of the present method can be compared with the calculated results of reference 5 for the $Pr = 0.733$ velocity profile. Figure 2 shows the neutral curve drawn in the α, Re -plane calculated by the present method and also shows some of the neutral points calculated in reference 5. Also shown in this figure is the point at which a finite disturbance was observed at $Re = 400$ and $\alpha = 0.367$ (ref. 10). The present method gives a minimum critical Reynolds number of 105 at $\alpha = 0.4$. This value compares well with the value of 103 calculated in reference 5. The lowest value of Re calculated in reference 2 for this case was $Re = 478$ at $\alpha = 2.54$. The values obtained from references 5, 10, and 2, respectively, were all converted to the values quoted, which correspond to the choice of reference quantities used herein.

In figure 3(a) the eigenfunctions ϕ_r , ϕ_i , ϕ_r' , and ϕ_i' are shown for the minimum critical Reynolds number of 105 at $\alpha = 0.4$ with $c_r = 0.1513$. Also shown in this figure are the locations of the critical layers. The qualitative agreement between these curves and the curves given in reference 5 for this case is good.

The distribution of the energy transfer functions throughout the boundary layer is also presented in figure 3(b), where the location of the critical layers is also shown. As is the case for ordinary boundary layers, most of the energy transfer from the basic flow by the Reynolds stress takes place at a critical layer. Of course, for ordinary boundary-layer velocity profiles there is only one critical layer. For the free-convection velocity profile, the outer critical layer appears to be more significant than the inner critical layer. As can be seen from figure 3(b), the dissipation is greatest near the plate. Of course, for neutral oscillations the net area under the two curves should add up to zero. Integration by Simpson's rule verified this for this case and for all cases to be discussed subsequently.

Stability Results for Water Without Temperature Fluctuations

The results for this case are qualitatively the same as for the previous case. Figure 4 shows the neutral curve drawn in the α, Re -plane calculated by the present method and also shows the points at which finite disturbances were observed in reference 1. These experimental points were obtained from figure 8 of reference 1. As can be seen from figure 4, all the experimental points lie within the region of amplification as obtained by the present method. The experimental points represent natural finite disturbances, which were observed without the use of some device to introduce controlled disturbances in order to

provoke the onset of turbulence. The lowest value of Re calculated in reference 1 for this case was $Re = 5040$ at $\alpha = 1.5$.

In figure 5(a) the eigenfunctions ϕ_r , ϕ_i , ϕ'_r , and ϕ'_i are shown for the minimum critical Reynolds number of 385 at $\alpha = 0.4125$ with $c_r = 0.0800$, and the distribution of energy transfer throughout the boundary layer is shown in figure 5(b). For the case of no temperature fluctuations, the plots of the eigenfunctions and the distribution of energy are quite similar for air and water.

Stability Results for Air With Temperature Fluctuations

In reference 1, it is indicated that for large values of αRe the temperature fluctuations can be neglected for the purpose of solving the eigenvalue problem. Of course, "large" is a relative term. For computational purposes, a large Reynolds number means that the asymptotic method may be applicable. In another sense, the statement that the Reynolds number is not too large can mean that the direct numerical method is applicable. Of course, the regions of applicability of these two methods could overlap. Since no numerical difficulties were being encountered in the stability calculations with no temperature fluctuations, it was decided that the Reynolds number was not too large. However, the task remained of determining whether or not the Reynolds number was large enough to neglect temperature fluctuations. For this reason, it was decided to examine the effects of temperature fluctuations.

Figure 6 shows the neutral curve drawn in the α, Re -plane for this case. It is seen that the neutral curve develops a hump for small values of α and Re and the minimum critical Reynolds number is lowered. The calculated point closest to the minimum critical point is at $Re = 64$, $\alpha = 0.15$ with $c_r = 0.2692$. Also shown in this figure is the neutral curve for no temperature fluctuations. If a path along the neutral curve for the case with temperature fluctuations is followed, proceeding to lower values of α below 0.15, it is seen that c_r increases continually until, at Re equal to about 70, the phase velocity of the disturbance wave is greater than the maximum velocity of the basic flow (see fig. 1). Consequently, there are no critical layers for values of α less than about 0.13. This result is impossible for ordinary boundary-layer flows for which the Orr-Sommerfeld equation is applicable (see ref. 11).

It is of interest to examine the eigenfunctions and energy distribution, while proceeding along the neutral curve to lower values of α with c_r increasing. Figures 7(a), 8(a), and 9(a) show the eigenfunctions, and figures 7(b), 8(b), and 9(b) show the energy distributions for $\alpha = 0.45$, 0.15, and 0.04, respectively.

Examination of figure 7(a) shows that the plots of the velocity eigenfunctions with temperature fluctuations resemble the plots of the velocity eigenfunctions for no temperature fluctuations (fig. 3(a)). Also shown in figure 7(a) are the temperature eigenfunctions. As the values of α become smaller, the plots of the velocity eigenfunctions change their shape, and the curves tend to oscillate less than at higher values of α (see figs. 7(a), 8(a), and 9(a)).

Examination of the energy distributions for $\alpha = 0.45$ (fig. 7(b)) shows that the outer critical layer is significant in that the peak of the Reynolds stress-term curve is located near it, as is the case without temperature fluctuations. It can be seen that the energy distribution is qualitatively the same as for no temperature fluctuations, except that the temperature fluctuation term is reinforcing the Reynolds stress term.

When the two critical layers are almost coincident at $\alpha = 0.15$, it is seen in figure 8(b) that the critical layers have no correlation with the energy peak but the buoyancy term is giving a positive contribution as in the previous case. Note that in this case the Reynolds stress term is not adding energy to the disturbance flow, but is actually subtracting energy from it. When the critical layer has completely disappeared as in figure 9(b), it is seen that the buoyancy term is still adding energy to the disturbance flow, and again the Reynolds stress term is subtracting.

From the previous results it appears that the neglect of temperature fluctuations for the purpose of solving the eigenvalue problem can be justified in the vicinity of $\alpha = 0.35$ to 0.55 , where the eigenvalues for the case without temperature fluctuations are close to the eigenvalues for this case. However, as can be seen from figure 7(a), the temperature fluctuations are not negligible even in this range of α .

For values of α smaller than $\alpha = 0.35$, the buoyancy term in the energy balance is significant in that it is the only term that is giving a positive contribution to the energy of the disturbance motion. For $\alpha = 0.15$ and 0.04 it can be seen that the Reynolds stress term is actually extracting energy. It appears that the introduction of temperature fluctuations introduces a new mode of instability. This new mode is characterized by the buoyancy force term in the energy balance assuming a dominant role. The buoyancy force term is independent of any property of the basic flow velocity profile and is proportional to the gravitational acceleration.

The presence of buoyancy effects allows the phase velocity of the disturbance wave to be greater than the maximum velocity of the basic flow. In other problems of hydrodynamic stability where the energy supply to the disturbance is proportional to the gravitational force, the phase velocity is greater than the maximum velocity of the basic flow. In fact, in reference 12 it is shown, for small wave numbers, that the phase velocity is equal to twice the maximum velocity for laminar flow down an inclined plane.

Stability Results for Water With Temperature Fluctuations

The amount of information obtained for this case is significantly less than for the previous case of air with temperature fluctuations, because numerical difficulties were encountered for large values of α and Re . Figure 10 shows the neutral curve drawn in the α, Re -plane. All the points on this curve represent eigenvalues which have the property that the phase velocity of the disturbance wave is greater than the maximum velocity of the basic flow. For air (fig. 6) this property was obtained only for $\alpha < 0.13$. Also shown in figure 10

is the neutral curve for water without temperature fluctuations. It was difficult to trace the curve with temperature fluctuations for higher values of α than shown, because the values of the eigenfunctions at the edge of the boundary layer changed markedly with each run even though the eigenvalues and starting values were only changing in the eighth decimal place. However, even though the numerical method cannot produce the eigenfunctions, the eigenvalues so obtained can be accepted as being reliable (see ref. 7). The point at $\alpha = 0.75$ and $Re = 385$ in figure 10 is one where eigenvalues, but not the eigenfunctions, can be obtained. This point is on a neutral curve with phase velocity $c_r = 0.1241$, which is less than the maximum velocity in the basic flow (see fig. 1).

The eigenfunctions for $Re = 34$ at $\alpha = 0.45$ with $c_r = 0.1556$ are shown in figure 11(a), and the energy distribution for this case is shown in figure 11(b). It can be seen that the two cases, air and water, with temperature fluctuations are qualitatively the same in that the buoyancy term is providing most of the energy input into the disturbance motion when the phase velocity is greater than the maximum velocity in the boundary layer.

CONCLUDING REMARKS

The direct numerical method gives a minimum critical Reynolds number lower than the Reynolds number at which finite disturbances were observed experimentally, whereas calculations based on asymptotic techniques yield a minimum critical Reynolds number higher than the Reynolds number at which finite disturbances were observed. However, the direct method is not universally applicable to all problems for all ranges of the parameters, especially when the Reynolds number is large. Generally, the method should give results in the lower left corner of the α, Re -plane where the minimum critical Reynolds number is usually found.

By referring to the results of experiments with natural disturbances, that is, experiments conducted without the use of controlled disturbances to provoke the onset of turbulence, it can be concluded that the minimum critical Reynolds number calculated herein for the two cases without temperature fluctuations provides a lower bound to the Reynolds number at which finite disturbances were observed. However, inclusion of temperature fluctuations indicated that this minimum critical Reynolds number is not the least lower bound. Rather, there is a lower minimum critical Reynolds number at small values of wave number α . This instability, for which the phase velocity of the disturbance wave is greater than the maximum velocity of the basic flow in the boundary layer, has not been reported in the accounts of natural transition experiments.

The existence of temperature fluctuations provides a new mechanism of energy transfer to the kinetic energy of the disturbance motion. This amount of energy is a significant contribution to the energy balance in that, for all cases calculated, it was always positive, that is, destabilizing.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, September 19, 1963

APPENDIX A

SYMBOLS

a	$\varphi'''(0)$
b	$s'(0)$
c	phase velocity
c_p	specific heat at constant pressure
e_B	buoyancy
e_D	dissipation
e_{Re}	Reynolds stress
F	stream function, dimensionless
Fr	Froude number
Gr_x	Grashof number based on x, $g\beta^* \Delta T x^3 / \nu_\infty^2$
g	gravitational acceleration
H	temperature dimensionless
i	imaginary unit
KE	defined in eq. (42)
k	coefficient of heat conductivity
P	pressure of basic flow
Pr	Prandtl number
p	pressure
Re	Reynolds number
\Re	real part
s	temperature amplitude function, dimensionless
T	temperature, basic flow
ΔT	$T_w - T_\infty$

T_w	wall temperature
T_∞	ambient temperature
t	temperature
U	velocity parallel to plate, basic flow
U^*	reference velocity parallel to plate
u	velocity parallel to plate
V	velocity normal to plate, basic flow
v	velocity normal to plate
x	distance from leading edge of plate
y	normal distance from plate
α	wave number
β^*	coefficient of volumetric expansion
δ	reference length, $\sqrt{2x}/(Gr_x)^{1/4}$
η	normal distance from plate, dimensionless
η_e	edge of boundary layer
λ	wavelength, $2\pi/\alpha$, dimensionless
μ	coefficient of viscosity
ν	kinematic viscosity
ξ	distance from leading edge of plate, dimensionless
ρ	density
τ	time, dimensionless
ϕ	stream function amplitude, dimensionless
ψ	stream function, dimensionless

Subscripts:

a, b, c	denotes differentiation with respect to quantity
i	refers to imaginary part

max maximum

r refers to real part

Superscripts:

~ disturbance quantity

- dimensional quantity

' differentiation with respect to η

APPENDIX B

DERIVATION OF STABILITY EQUATIONS

The governing equations for a parallel basic flow plus a disturbance as given in reference 1 are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (B1)$$

$$\frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u + g\beta^*(t - t_\infty) \quad (B2)$$

$$\frac{\partial v}{\partial \tau} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 v \quad (B3)$$

$$\frac{\partial t}{\partial \tau} + u \frac{\partial t}{\partial x} + v \frac{\partial t}{\partial y} = \frac{k}{\rho c_p} \nabla^2 t \quad (B4)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The basic steady-state flow is the free-convection flow about a vertical heated plate. In the derivation of the governing equations the variation of the density has been neglected except in the gravity term where the density was taken to depend linearly on the temperature. In equations (B1) to (B4) the density, where it appears, is taken to be a constant. In accordance with the parallel-flow assumption, the basic flow can be described by $U = U(y)$, $V = 0$, $P = P(y)$, and $T = T(y)$. Superimposed upon the basic flow is a two-dimensional disturbance. The disturbance equations are obtained by substituting $u = U + \tilde{u}$, $v = \tilde{v}$, $p = P + \tilde{p}$, and $t = T + \tilde{t}$ in the previous equations, subtracting out the basic flow, and neglecting products of disturbance quantities and their derivatives. The disturbance equations are

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0 \quad (B5)$$

$$\frac{\partial \tilde{u}}{\partial \tau} + U \frac{\partial \tilde{u}}{\partial x} + \tilde{v} \frac{\partial U}{\partial y} = - \frac{1}{\rho} \frac{\partial \tilde{p}}{\partial x} + \nu \nabla^2 \tilde{u} + g\beta \tilde{t} \quad (B6)$$

$$\frac{\partial \tilde{v}}{\partial \tau} + U \frac{\partial \tilde{v}}{\partial x} = - \frac{1}{\rho} \frac{\partial \tilde{p}}{\partial y} + \nu \nabla^2 \tilde{v} \quad (B7)$$

$$\frac{\partial \tilde{t}}{\partial \tau} + U \frac{\partial \tilde{t}}{\partial x} + \tilde{v} \frac{dT}{dy} = \frac{k}{\rho c_p} \nabla^2 \tilde{t} \quad (B8)$$

The disturbance velocities can be obtained from a stream function

$$\bar{\psi}(x, y, \bar{\tau}) = \bar{\phi}(y) \exp \left[i\bar{\alpha}(x - \bar{c}\bar{\tau}) \right] \quad (\text{B9})$$

from which

$$\tilde{u} = \frac{\partial \bar{\psi}}{\partial y} = \bar{\phi}'(y) \exp \left[i\bar{\alpha}(x - \bar{c}\bar{\tau}) \right] \quad (\text{B10})$$

$$\tilde{v} = - \frac{\partial \bar{\psi}}{\partial x} = - i\bar{\alpha}\bar{\phi}(y) \exp \left[i\bar{\alpha}(x - \bar{c}\bar{\tau}) \right] \quad (\text{B11})$$

As can be seen from equation (B9), the disturbance is taken to be periodic in the distance x along the plate. The positive quantity $\bar{\alpha}$ is the wave number of a disturbance wave, and \bar{c}_r , the real part of \bar{c} , is the velocity of propagation of the wave. The imaginary part of \bar{c} will determine whether the disturbance will grow ($\bar{c}_i > 0$) or decay ($\bar{c}_i < 0$) in time. The temperature disturbance is also periodic in the distance x along the plate and may be expressed as

$$\tilde{t}(x, y, \bar{\tau}) = \bar{s}(y) \exp \left[i\bar{\alpha}(x - \bar{c}\bar{\tau}) \right] \quad (\text{B12})$$

It is convenient to represent the disturbance quantities in complex form in order to satisfy the phase relations imposed by equations (B5) to (B8). However, physical significance is to be attached only to the real part of disturbance quantities.

Dimensionless variables are introduced by choosing reference quantities in conformity with those used in calculating the basic flow as follows:

Length:

$$\frac{\sqrt{2x}}{(\text{Gr}_x)^{1/4}} \equiv \delta$$

Velocity:

$$\frac{2\nu(\text{Gr}_x)^{1/2}}{x} \equiv U^*$$

Temperature:

$$T_w - T_\infty \equiv \Delta T$$

where

$$\text{Gr}_x = \frac{g\beta^* \Delta T x^3}{\nu_\infty^2}$$

The following dimensionless variables are used to transform equations (B5) to (B8) into the dimensionless form:

$$x = \xi \delta \quad (B13)$$

$$y = \eta \delta \quad (B14)$$

$$\tau = \frac{\tau \delta}{U^*} \quad (B15)$$

$$U = F' U^* \quad (B16)$$

$$T - T_\infty = H \Delta T \quad (B17)$$

$$\bar{\varphi} = \varphi \delta U^* \quad (B18)$$

$$\bar{s} = s \Delta T \quad (B19)$$

$$\bar{c} = c U^* \quad (B20)$$

$$\bar{\alpha} = \frac{\alpha}{\delta} \quad (B21)$$

In terms of the new dimensionless variables, the stability equations are obtained by substitution of equations (B10) and (B11) into equations (B5) to (B8); the result is

$$(F' - c)(\varphi'' - \alpha^2 \varphi) - F''' \varphi = - \frac{1}{\alpha Re} (\varphi'''' - 2\alpha^2 \varphi'' + \alpha^4 \varphi) - \frac{1\beta^* \Delta T}{\alpha Fr} s' \quad (B22)$$

$$(F' - c)s - \varphi H' = - \frac{1}{\alpha Re Pr} (s'' - \alpha^2 s) \quad (B23)$$

where the Reynolds number Re is defined as $Re = 8U^*/\nu = 2\sqrt{2}(Gr_x)^{1/4}$, the Froude number is $Fr = (U^*)^2/g\delta = 4\nu_\infty^2 Gr_x/g\delta x^2$, and the Prandtl number is $Pr = \mu c_p/k$. Equations (B22) and (B23) become identical to equations (1) and (2), respectively, when the substitution $Fr/\beta^* \Delta T = Re$ is made.

Equations (B22) and (B23) differ from the ones given in reference 2 in that the term s' in equation (B22) is multiplied by $F_\infty^2/(F'_{\max})^3$. This difference is due to the choice of reference quantities in reference 2, which differ from those of reference 1 used hereinafter. The reference quantities of reference 2 are as follows:

Length:

$$\frac{F_\infty}{F'_{\max}} \frac{\sqrt{2}x}{(Gr_x)^{1/4}}$$

Velocity:

$$F'_{max} \left[\frac{2v_{\infty} (Gr_x)^{1/2}}{x} \right]$$

APPENDIX C

CONDITIONS TO BE SATISFIED BY EXPONENTIALLY DECAYING SOLUTIONS

The technique used in reference 4 is used to eliminate the arbitrary constants in equations (7) and (8) in order to obtain a set of linear and homogeneous relations in the dependent variables of equations (1) and (2) and their derivatives, which are to be satisfied at the edge of the boundary layer.

In order to satisfy the boundary conditions (eq. (4)), it is necessary that $c_1 = c_3 = c_5 = 0$. Further, in order that the solutions (7) and (8) agree with the numerical solution of equations (1) and (2), which is obtained by numerical integration when $\eta = \eta_e$, it is necessary that

$$\begin{aligned} \varphi = & c_1 \exp(\alpha\eta_e) + c_2 \exp(-\alpha\eta_e) + c_3 \exp(\beta\eta_e) + c_4 \exp(-\beta\eta_e) \\ & + c_5 \exp(\gamma\eta_e) + c_6 \exp(-\gamma\eta_e) \end{aligned} \quad (C1)$$

$$\begin{aligned} \varphi' = & c_1 \alpha \exp(\alpha\eta_e) - c_2 \alpha \exp(-\alpha\eta_e) + c_3 \beta \exp(\beta\eta_e) - c_4 \beta \exp(-\beta\eta_e) \\ & + c_5 \gamma \exp(\gamma\eta_e) - c_6 \gamma \exp(-\gamma\eta_e) \end{aligned} \quad (C2)$$

$$\begin{aligned} \varphi'' = & c_1 \alpha^2 \exp(\alpha\eta_e) + c_2 \alpha^2 \exp(-\alpha\eta_e) + c_3 \beta^2 \exp(\beta\eta_e) + c_4 \beta^2 \exp(-\beta\eta_e) \\ & + c_5 \gamma^2 \exp(\gamma\eta_e) + c_6 \gamma^2 \exp(-\gamma\eta_e) \end{aligned} \quad (C3)$$

$$\begin{aligned} \varphi''' = & c_1 \alpha^3 \exp(\alpha\eta_e) - c_2 \alpha^3 \exp(-\alpha\eta_e) + c_3 \beta^3 \exp(\beta\eta_e) - c_4 \beta^3 \exp(-\beta\eta_e) \\ & + c_5 \gamma^3 \exp(\gamma\eta_e) - c_6 \gamma^3 \exp(-\gamma\eta_e) \end{aligned} \quad (C4)$$

$$s = -c_5 \frac{(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2) \exp(\gamma\eta_e)}{\gamma} + c_6 \frac{(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2) \exp(-\gamma\eta_e)}{\gamma} \quad (C5)$$

$$s' = -c_5 (\gamma^2 - \alpha^2)(\gamma^2 - \beta^2) \exp(\gamma\eta_e) - c_6 (\gamma^2 - \alpha^2)(\gamma^2 - \beta^2) \exp(-\gamma\eta_e) \quad (C6)$$

where the left sides of equations (C1) to (C6) are obtained from the numerical integration of equations (1) and (2) from $\eta = 0$ to $\eta = \eta_e$. Solution of equations (C1) to (C6) gives the c 's. Since $c_1 = c_3 = c_5 = 0$, the three determinants that give c_1 , c_3 , and c_5 must vanish. Hence,

$$\begin{vmatrix}
\varphi & 1 & 1 & 1 & 1 & 1 \\
\varphi' & -\alpha & \beta & -\beta & \gamma & -\gamma \\
\varphi'' & \alpha^2 & \beta^2 & \beta^2 & \gamma^2 & \gamma^2 \\
\varphi''' & -\alpha^3 & \beta^3 & -\beta^3 & \gamma^3 & -\gamma^3 \\
s & 0 & 0 & 0 & -\frac{(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)}{\gamma} & \frac{(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)}{\gamma} \\
s' & 0 & 0 & 0 & -(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2) & -(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)
\end{vmatrix} = 0 \quad (C7)$$

$$\begin{vmatrix}
1 & 1 & \varphi & 1 & 1 & 1 \\
\alpha & -\alpha & \varphi' & -\beta & \gamma & -\gamma \\
\alpha & \alpha^2 & \varphi'' & \beta^2 & \gamma^2 & \gamma^2 \\
\alpha^3 & -\alpha^3 & \varphi''' & -\beta^3 & \gamma^3 & -\gamma^3 \\
0 & 0 & s & 0 & -\frac{(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)}{\gamma} & \frac{(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)}{\gamma} \\
0 & 0 & s' & 0 & -(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2) & -(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)
\end{vmatrix} = 0 \quad (C8)$$

$$\begin{vmatrix}
1 & 1 & 1 & 1 & \varphi & 1 \\
\alpha & -\alpha & \beta & -\beta & \varphi' & -\gamma \\
\alpha^2 & \alpha^2 & \beta^2 & \beta^2 & \varphi'' & \gamma^2 \\
\alpha^3 & -\alpha^3 & \beta^3 & -\beta^3 & \varphi''' & -\gamma^3 \\
0 & 0 & 0 & 0 & s & \frac{(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)}{\gamma}
\end{vmatrix} = 0 \quad (C9)$$

Equations (C7) to (C9) reduce to

$$(\gamma^2 - \alpha^2)[\varphi''' + \alpha\varphi'' - \beta^2(\varphi' + \alpha\varphi)] + \gamma^2 s + \alpha s' = 0 \quad (C10)$$

$$(\gamma^2 - \beta^2)[\varphi''' + \beta\varphi'' - \alpha^2(\varphi' + \beta\varphi)] + \gamma^2 s + \beta s' = 0 \quad (C11)$$

$$s' + \gamma s = 0 \quad (C12)$$

Equations (C10) to (C12) are the conditions that the numerical solution must satisfy at the edge of the boundary layer at $\eta = \eta_e$. It is convenient to eliminate s' from equations (C10) and (C11) by means of equation (C12). This leads to the somewhat simpler conditions that will be employed instead of equations (C11) and (C10), respectively, namely,

$$\varphi''' - \alpha^2 \varphi' + \beta(\varphi'' - \alpha^2 \varphi) + \frac{\gamma}{\gamma + \beta} s = 0 \quad (C13)$$

$$\varphi''' + \alpha \varphi'' - \beta^2(\varphi' - \alpha \varphi) + \frac{\gamma}{\gamma + \alpha} s = 0 \quad (C14)$$

It is clear that, if equations (C12) to (C14) are satisfied, then so are equations (C10) and (C11).

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TABLE I. - FUNCTIONS F AND H AND DERIVATIVES
FOR PRANDTL NUMBER OF 6.7

η	F	F'	F''	F'''	H	H'
0.	0.	0.	0.4547	-1.0000	1.0000	-1.0408
.1250	.0032	.0494	.3379	-.8684	.8700	-1.0379
.2500	.0118	.0852	.2377	-.7351	.7412	-1.0195
.3750	.0241	.1095	.1541	-.6033	.6162	-.9752
.5000	.0388	.1244	.0866	-.4777	.4986	-.9016
.6250	.0548	.1318	.0342	-.3628	.3919	-.8017
.7500	.0715	.1335	-.0047	-.2623	.2989	-.6841
.8750	.0880	.1311	-.0320	-.1783	.2212	-.5599
1.0000	.1041	.1259	-.0500	-.1115	.1588	-.4397
1.1250	.1194	.1189	-.0606	-.0607	.1107	-.3320
1.2500	.1338	.1110	-.0657	-.0240	.0751	-.2415
1.3750	.1472	.1026	-.0671	.0011	.0496	-.1696
1.5000	.1595	.0943	-.0658	.0174	.0319	-.1154
1.6250	.1707	.0862	-.0630	.0271	.0201	-.0762
1.7500	.1810	.0786	-.0593	.0322	.0124	-.0490
1.8750	.1904	.0714	-.0551	.0342	.0075	-.0307
2.0000	.1989	.0648	-.0508	.0343	.0044	-.0188
2.1250	.2066	.0587	-.0466	.0332	.0026	-.0113
2.2500	.2136	.0532	-.0425	.0314	.0015	-.0067
2.3750	.2200	.0481	-.0387	.0293	.0008	-.0039
2.5000	.2257	.0435	-.0352	.0271	.0005	-.0022
2.6250	.2309	.0393	-.0319	.0249	.0003	-.0012
2.7500	.2355	.0355	-.0289	.0228	.0001	-.0007
2.8750	.2397	.0321	-.0262	.0208	.0001	-.0004
3.0000	.2436	.0289	-.0237	.0190	.0000	-.0002
3.1250	.2470	.0261	-.0215	.0172	.0000	-.0001
3.2500	.2501	.0236	-.0194	.0157	.0000	-.0001
3.3750	.2529	.0213	-.0175	.0142	.0000	.0000
3.5000	.2554	.0192	-.0159	.0129	.0000	.0000
3.6250	.2577	.0173	-.0143	.0117	.0000	.0000
3.7500	.2597	.0156	-.0129	.0106	.0000	.0000
3.8750	.2616	.0140	-.0117	.0096	.0000	.0000
4.0000	.2633	.0127	-.0105	.0086	.0000	.0000
4.1250	.2648	.0114	-.0095	.0078	.0000	.0000
4.2500	.2661	.0103	-.0086	.0071	.0000	.0000
4.3750	.2673	.0093	-.0077	.0064	.0000	.0000
4.5000	.2684	.0083	-.0070	.0058	.0000	.0000
4.6250	.2694	.0075	-.0063	.0052	.0000	.0000
4.7500	.2703	.0068	-.0057	.0047	.0000	.0000
4.8750	.2711	.0061	-.0051	.0042	.0000	.0000
5.0000	.2718	.0055	-.0046	.0038	.0000	.0000
5.1250	.2725	.0049	-.0042	.0035	.0000	.0000
5.2500	.2731	.0044	-.0038	.0031	.0000	.0000
5.3750	.2736	.0040	-.0034	.0028	.0000	.0000
5.5000	.2741	.0036	-.0031	.0025	.0000	.0000
5.6250	.2745	.0032	-.0028	.0023	.0000	.0000
5.7500	.2749	.0029	-.0025	.0021	.0000	.0000
5.8750	.2752	.0026	-.0022	.0019	.0000	.0000
6.0000	.2755	.0023	-.0020	.0017	.0000	.0000
6.1250	.2758	.0021	-.0018	.0015	.0000	.0000
6.2500	.2760	.0019	-.0016	.0014	.0000	.0000
6.3750	.2763	.0017	-.0015	.0012	.0000	.0000
6.5000	.2765	.0015	-.0013	.0011	.0000	.0000
6.6250	.2766	.0013	-.0012	.0010	.0000	.0000
6.7500	.2768	.0012	-.0011	.0009	.0000	.0000
6.8750	.2769	.0011	-.0010	.0008	.0000	.0000
7.0000	.2771	.0010	-.0009	.0007	.0000	.0000
7.1250	.2772	.0009	-.0008	.0007	.0000	.0000
7.2500	.2773	.0008	-.0007	.0006	.0000	.0000
7.3750	.2774	.0007	-.0006	.0005	.0000	.0000
7.5000	.2775	.0006	-.0006	.0005	.0000	.0000
7.6250	.2775	.0005	-.0005	.0004	.0000	.0000
7.7500	.2776	.0005	-.0005	.0004	.0000	.0000
7.8750	.2776	.0004	-.0004	.0004	.0000	.0000
8.0000	.2777	.0004	-.0004	.0003	.0000	.0000
8.1250	.2777	.0003	-.0003	.0003	.0000	.0000
8.2500	.2778	.0003	-.0003	.0003	.0000	.0000
8.3750	.2778	.0002	-.0003	.0002	.0000	.0000
8.5000	.2778	.0002	-.0003	.0002	.0000	.0000

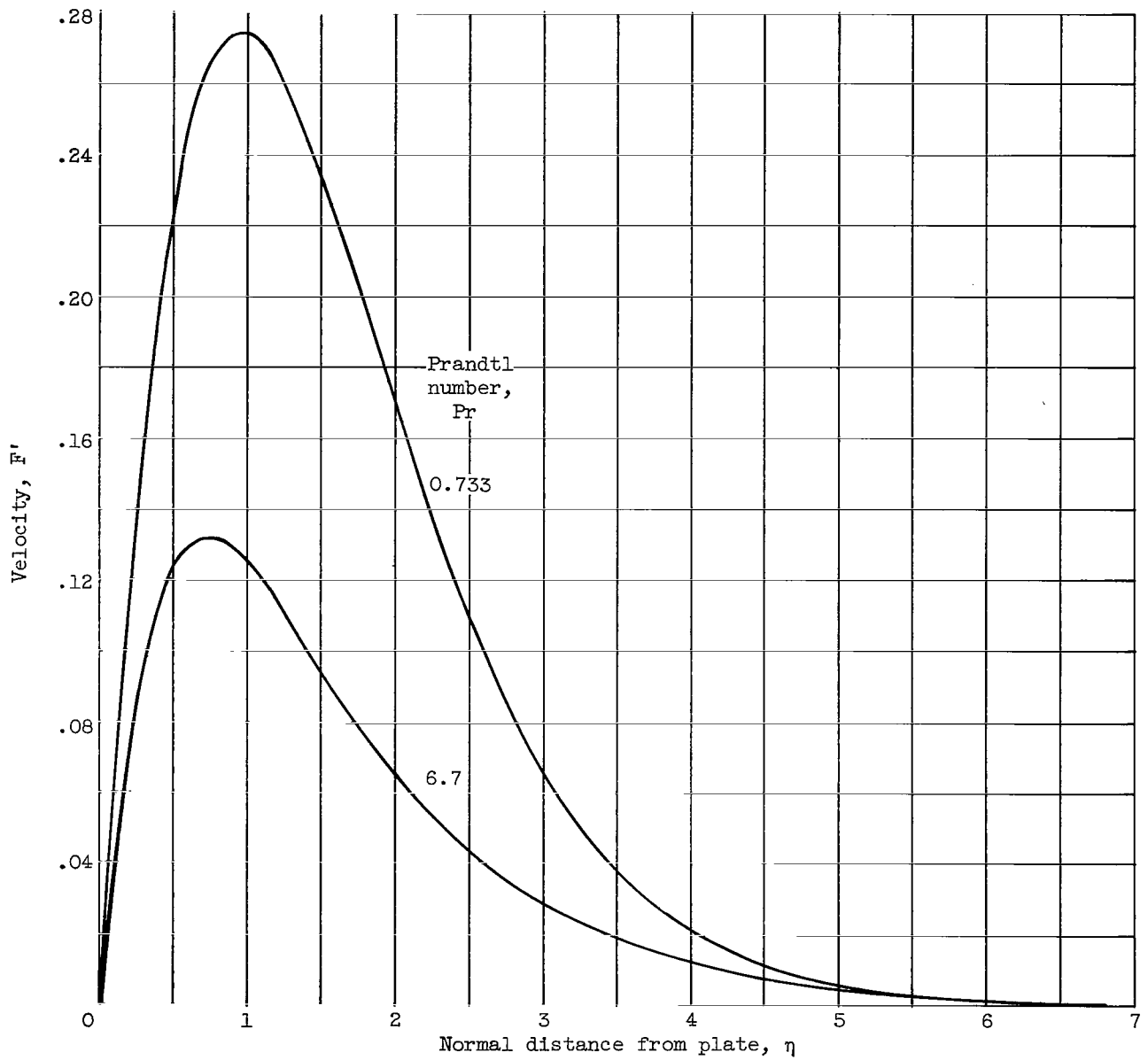


Figure 1. - Velocity profiles for two Prandtl numbers.

Figure 2. - Neutral curve for air without temperature fluctuations.

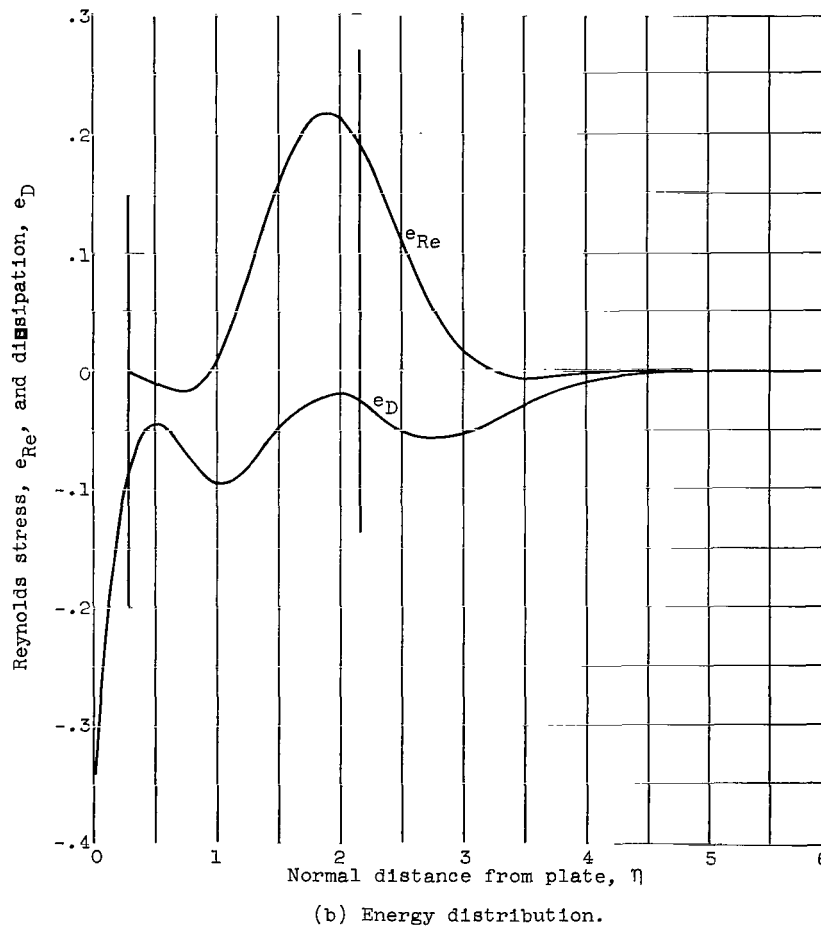
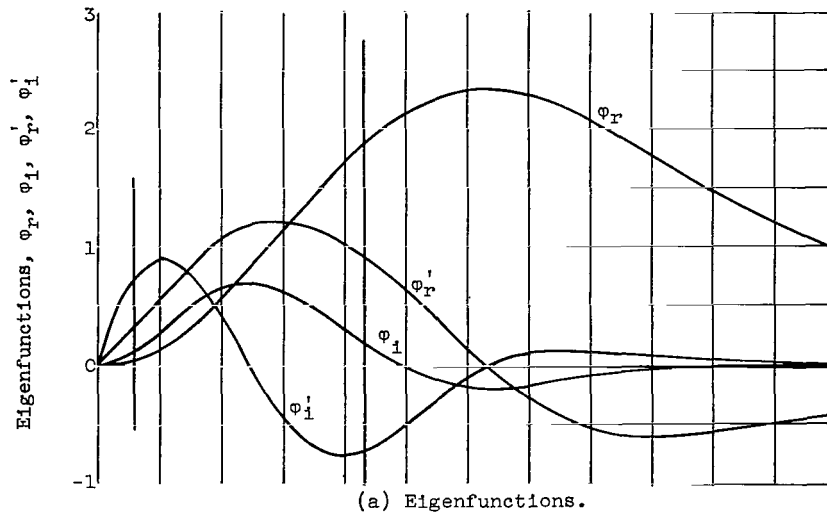


Figure 3. - Eigenfunctions and energy distribution for air without temperature fluctuations. Wave number α , 0.4; Reynolds number Re , 105; phase velocity c_r , 0.1513; phase velocity c_1 , 0. Vertical lines indicate location of critical layers.

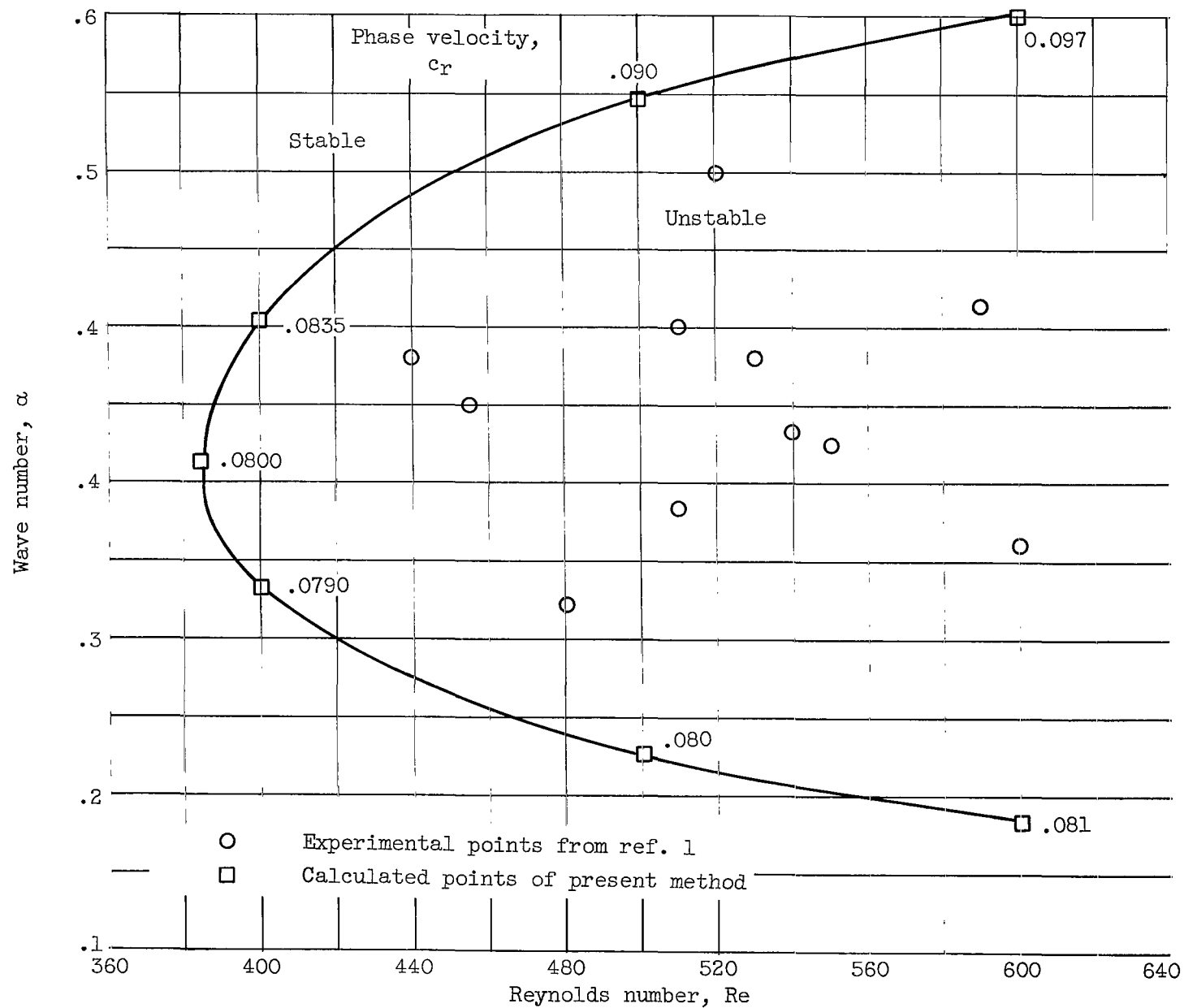


Figure 4. - Neutral curve for water without temperature fluctuations.

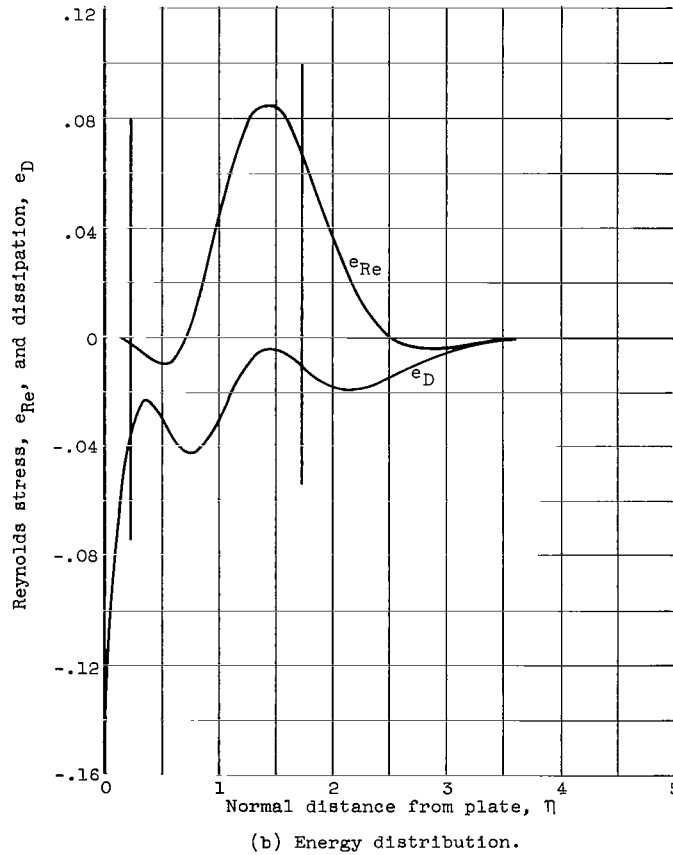
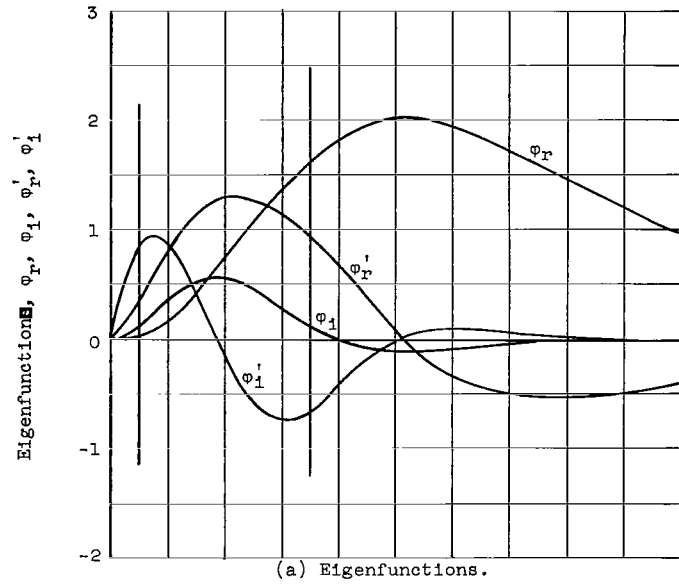


Figure 5. - Eigenfunctions and energy distribution for water without temperature fluctuations. Wave number α , 0.4125; Reynolds number Re , 385; phase velocity c_r , 0.800; phase velocity c_1 , 0. Vertical lines indicate location of critical layers.

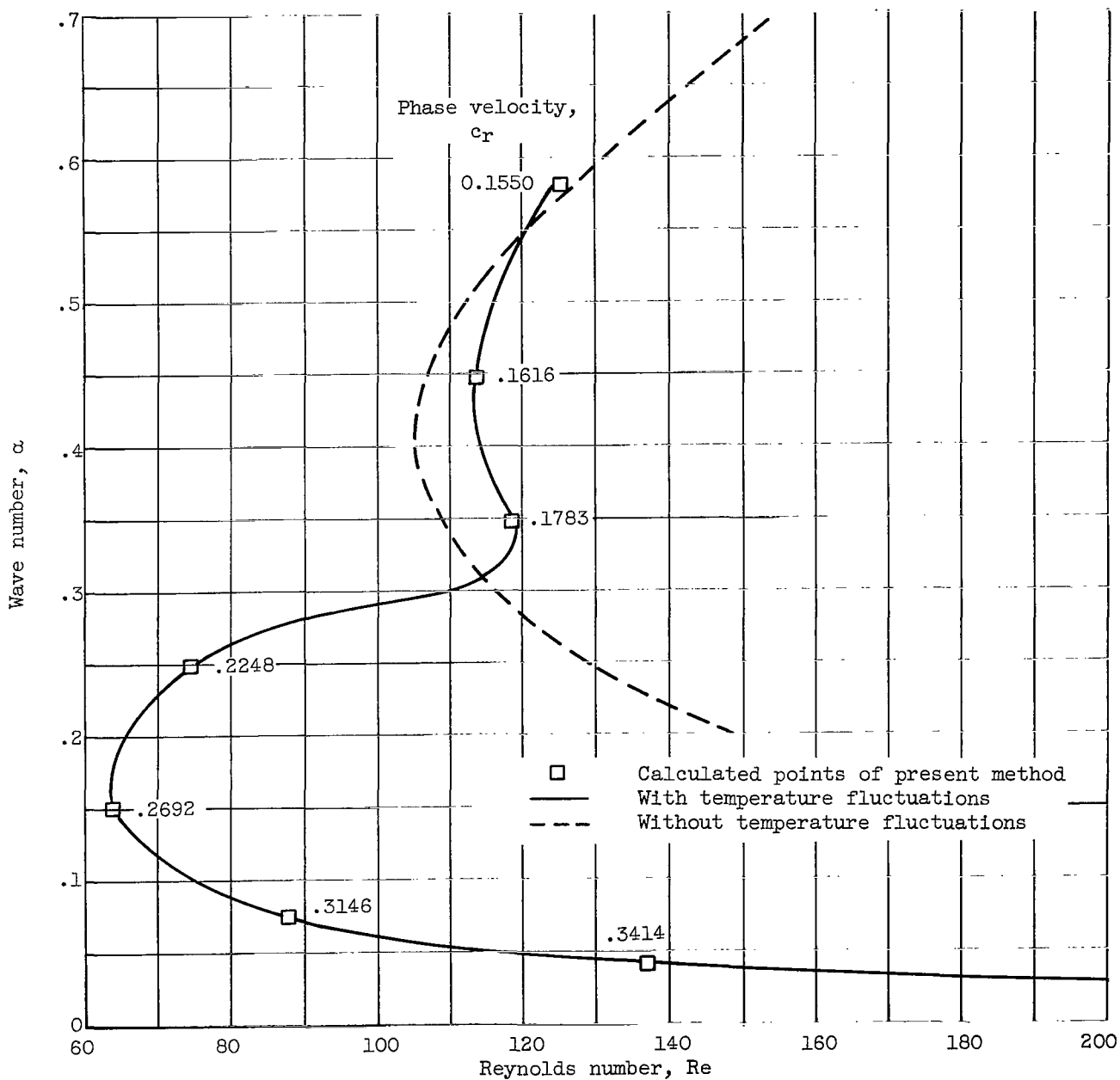


Figure 6. - Neutral curve for air with and without temperature fluctuations.

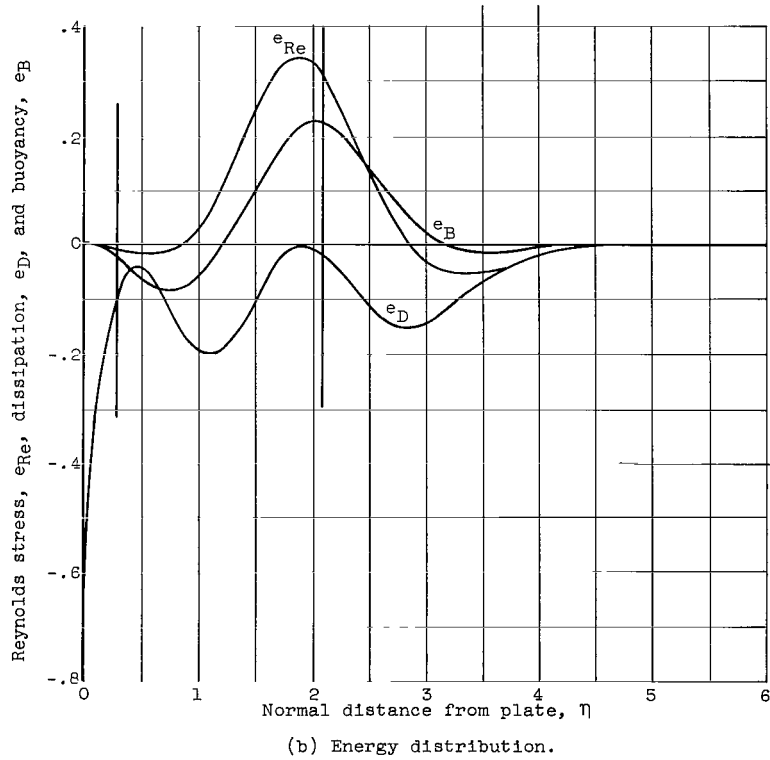
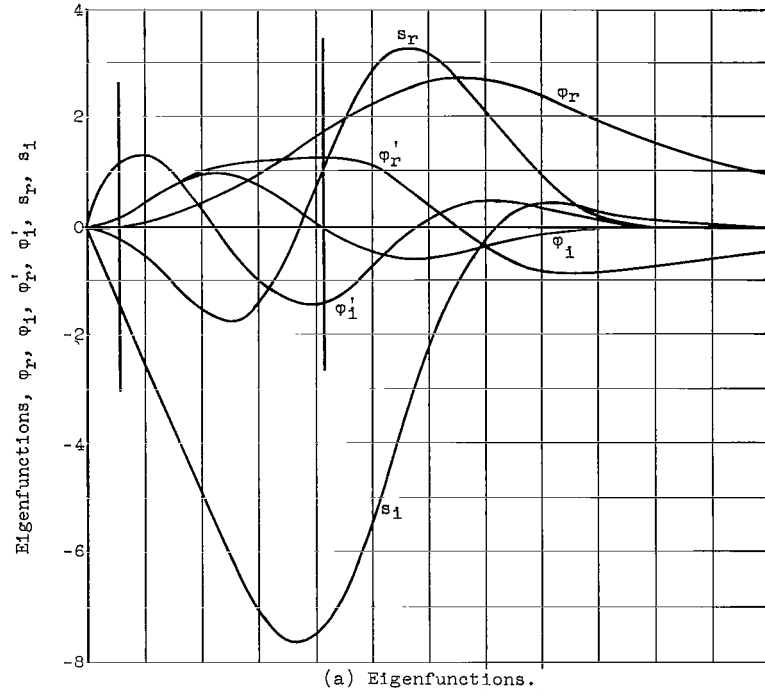


Figure 7. - Eigenfunctions and energy distribution for air with temperature fluctuations. Wave number α , 0.45; Reynolds number Re , 114; phase velocity c_r , 0.1616; phase velocity c_1 , 0.0002. Vertical lines indicate location of critical layers.

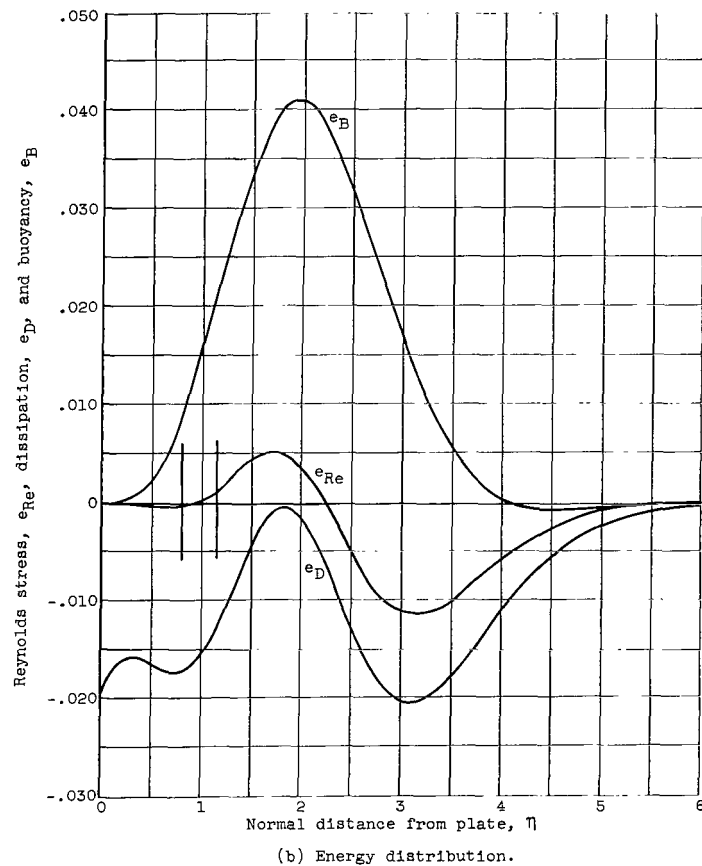
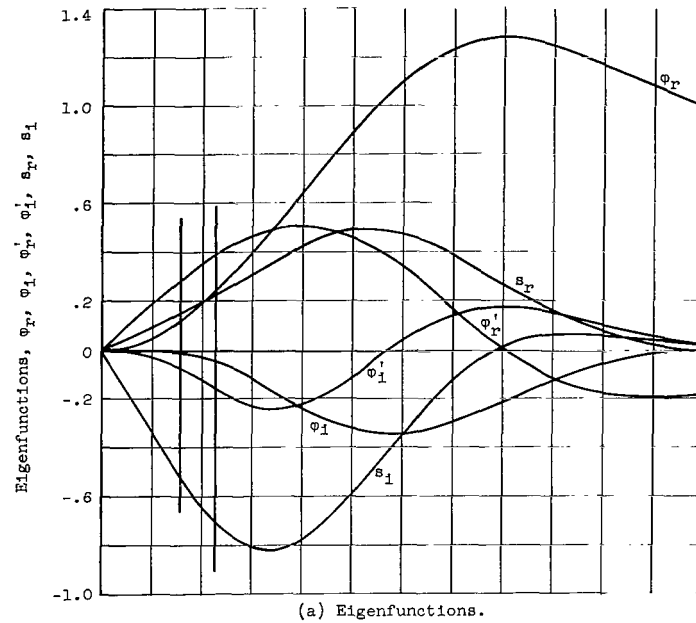
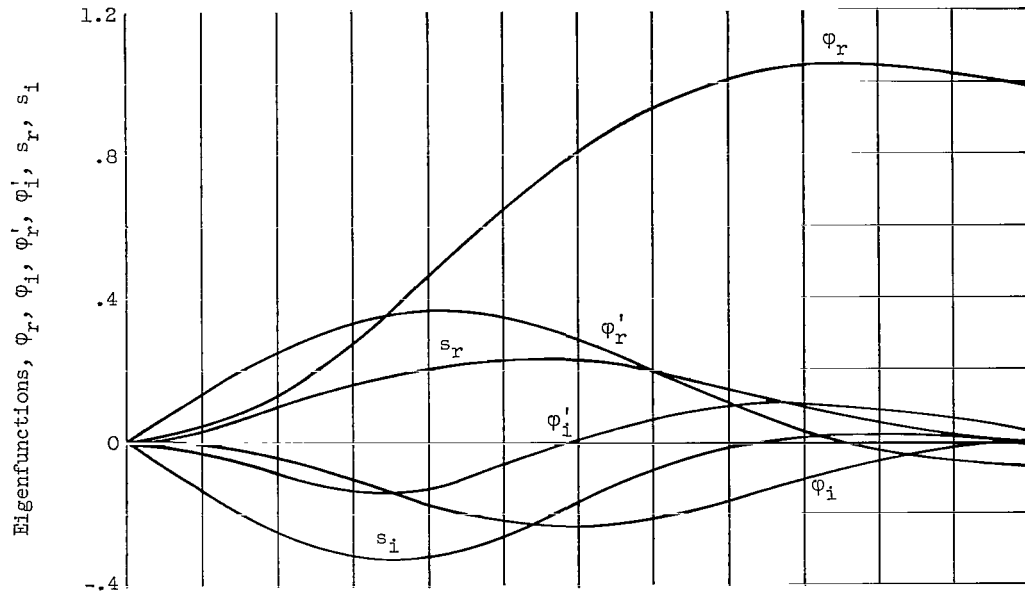
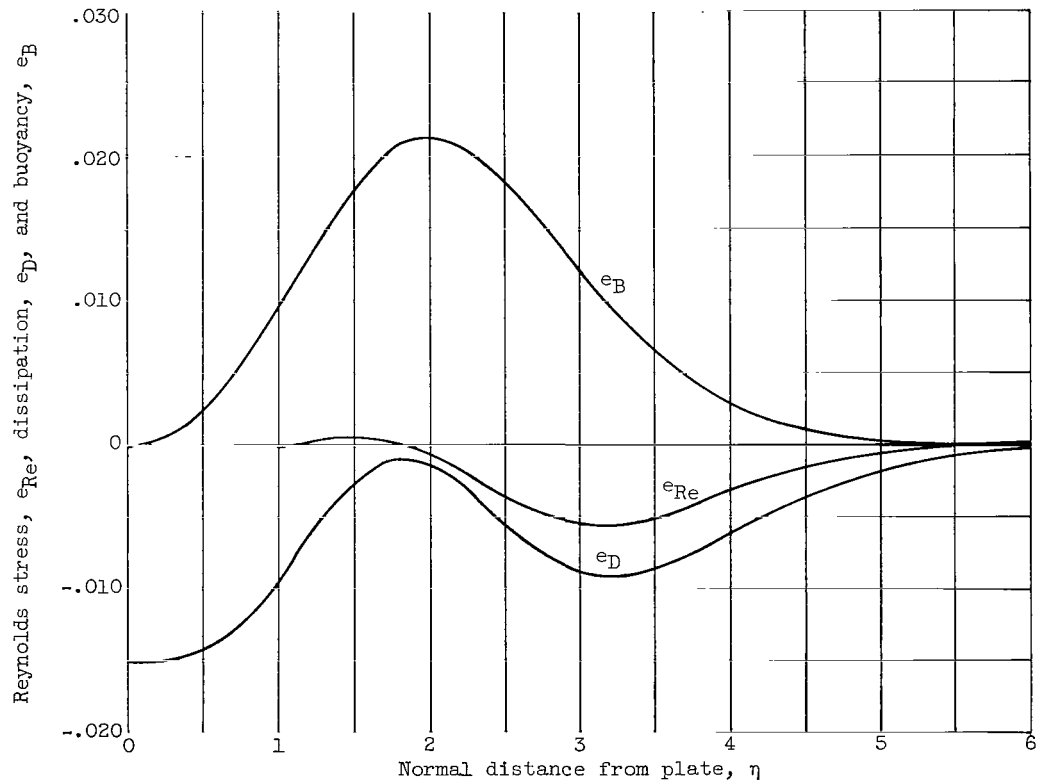


Figure 8. - Eigenfunctions and energy distribution for air with temperature fluctuations. Wave number α , 0.15; Reynolds number Re , 64; phase velocity c_r , 0.2692; phase velocity c_i , 0.0001. Vertical lines indicate location of critical layers.



(a) Eigenfunctions.



(b) Energy distribution.

Figure 9. - Eigenfunctions and energy distribution for air with temperature fluctuations. Wave number α , 0.04; Reynolds number Re , 137; phase velocity c_r , 0.3414; phase velocity c_i , 0.0001.

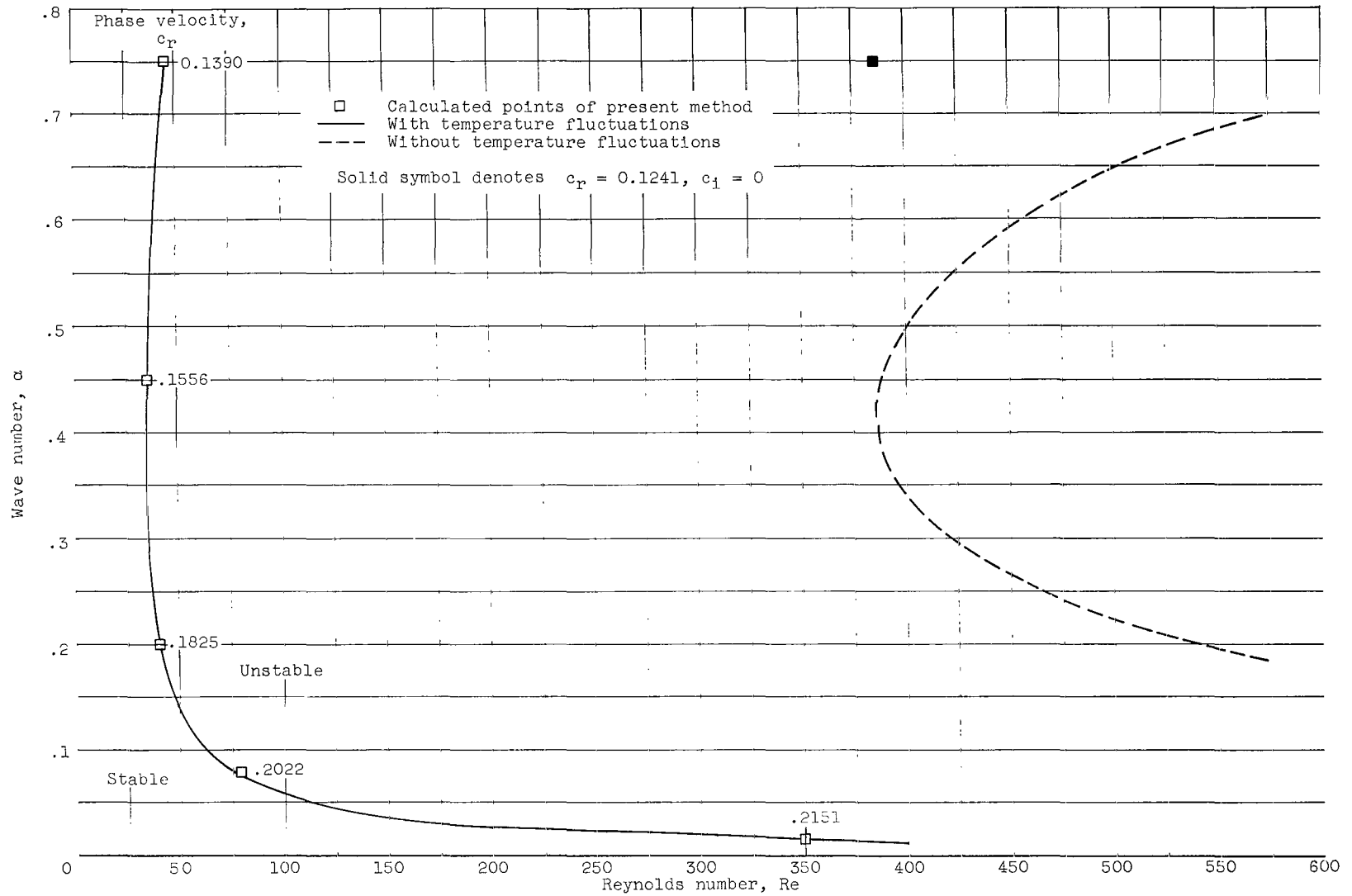


Figure 10. - Neutral curve for water with and without temperature fluctuations.

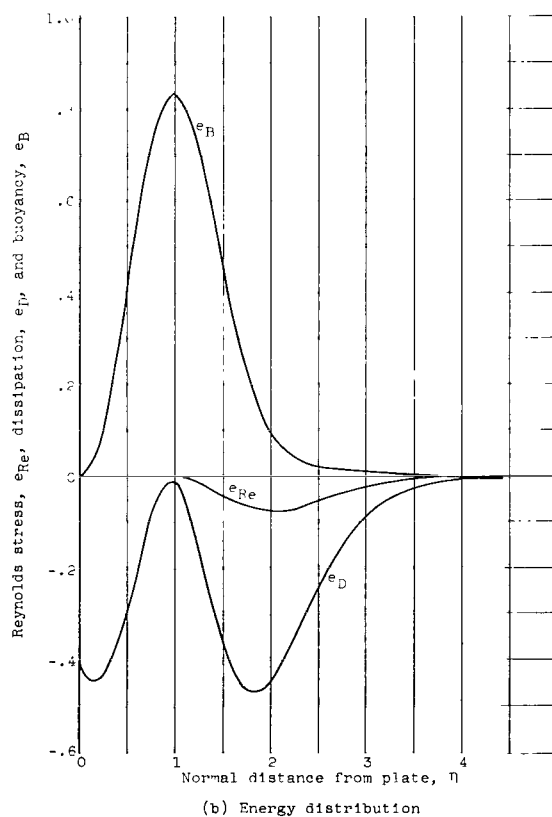
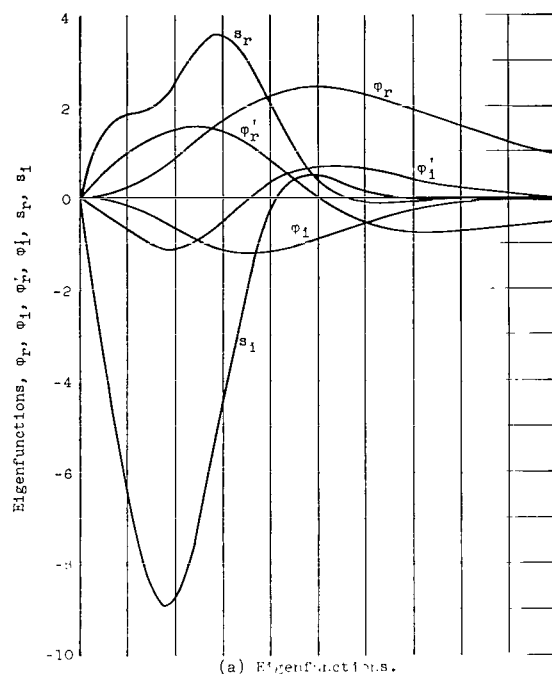


Figure 11. - Eigenfunctions and energy distribution for water with temperature fluctuations. Wave number α , 0.45; Reynolds number Re , 34; phase velocity c_r , 0.1556; phase velocity c_1 , 0.